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PARTIALLY OBSERVED STOCHASTIC CONTROL SYSTEMS

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1. Introduction.

In this paper we are concerned with the existence of optimal controls for problems of the following kind. Let X_t denote the process which we wish to control, Y_t the observation process and U_t the control process, $0 \le t \le T$, with T fixed. The state and observation processes are governed by stochastic differential equations

$$\begin{aligned} &(a)\quad dX_t = b(t,X_t,Y_t,U_t)dt + \sigma(t,X_t,Y_t)dW_t \\ &(b)\quad dY_t = h(t,X_t)dt + d\tilde{W}_t . \end{aligned} \tag{1.1}$$

X_t has values in N-dimensional R^N, Y_t values in R^1, and U_t values in $\mathcal{U} \subset R^\nu$. [Only some notational complications are involved if vector-valued observations Y_t are considered.] X_0 has given distribution, with density $p_0(x)$, and $Y_0 = 0$. In (1.1), W_t and $\tilde{W}_t$ are independent Wiener processes.

The problem is to minimize a criterion of the form

$$J = E\left\{\int_0^T F(t,X_t,U_t)dt + G(X_T)\right\}. \tag{1.2}$$

It is customary to require that U_t be measurable with respect to the σ-algebra generated by observations Y_s, $0 \le s \le t$. We call this the <u>strict sense</u> version of the problem. For several years the question of proving a general theorem about existence of optimal controls in the strict sense has been open. We do not obtain such a result here. However, we obtain an existence theorem in which a somewhat wider class of control processes is admitted. Roughly speaking, this wider class of controls is obtained as follows. Let

$$Z_t = \exp\left[\int_0^t h(s,X_s)dY_s - \frac{1}{2}\int_0^t h^2(s,X_s)ds\right]. \tag{1.3}$$

Then W_t, Y_t are independent Wiener processes under a new probability measure $\overset{\circ}{P}$ related to the original probability measure P by $\frac{dP}{d\overset{\circ}{P}} = -Z_T$. In the wide sense formulation we wish to require merely that U_s for $s \le t$ be independent of future increments $Y_r - Y_\rho$ for $t \le \rho < r$ with respect to $\overset{\circ}{P}$. In §2 we give a precise formulation of this idea, in which we define the control as the joint distribution measure of the processes Y, U.

Our method depends on introducing another stochastic control problem, which we call a "separated" problem. This separated problem is equivalent to the one formulated in §2. In the separated problem the "state" $p(t,\cdot)$ at time t is a function obeying a linear, parabolic partial differential equation (3.4). The coefficients of (3.4) depend on the observations Y_t and controls U_t, $0 \le t \le T$. The solution $p(t,x)$ is related in a simple way to the unnormalized conditional density $q(t,x)$ of X_t, given observations Y_s and controls U_s for $s \le t$. See (3.6). The proof of this fact makes use of probabilistic solutions to a "backward" partial differential equation adjoint to the "forward" equation (3.4), an idea already exploited in [3] for the nonlinear filter problem. However, unlike [3] we work with (3.4) instead of the Zakai equation (3.7) for q. In this way, Itô stochastic integrals and results about stochastic PDE's are avoided in the analysis. For the nonlinear filter problem, equation (3.4) was derived by Davis [1].

2. Formulation of the problem.

We make the following assumptions about the functions b, σ, h in (1.1).

(A_1) σ and its partial derivatives $\partial\sigma/\partial x_j$, $j = 1,\dots,N$, are bounded, continuous functions of (t,x,y). Moreover, σ has an inverse σ^{-1}, which is a bounded function of (t,x,y).

(A_2) $b(t,x,y,u) = b^0(t,x,y) + ub^1(t,x,y)$, where b^0 and b^1 are bounded, continuous functions of (t,x,y).

(A_3) $h, \partial h/\partial t, \partial h/\partial x_i, \partial^2 h/\partial x_i \partial x_j$, $i, j = 1,\dots,N$ are bounded, continuous functions.

We also assume:

(A_4) $\mathcal{U}$ is a convex, compact subset of R^ν.

(A_5) The density $p_0(x)$ of X_0 is in $L^2(R^N)$, and $\int_{R^N} |x|^\ell p_0(x)dx < \infty$ for some $\ell \ge 1$.

We formulate the problem on the "canonical" sample space

$$\Omega = C([0,T];R^N) \times C([0,T];R^1) \times L^2([0,T];\mathcal{U}),$$

whose elements ω satisfy

$$\omega(t) = (X_t(\omega), Y_t(\omega), U_t(\omega)), \quad 0 \le t \le T.$$

We give $C([0,T];R^d)$ for $d = 1, N$ the usual norm topology; and we give $L^2([0,T];\mathcal{U})$ the weak topology, which is metrizable since $\mathcal{U}$ is compact. We consider the following increasing families of σ-algebras:

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$$\mathscr{F}_t = \sigma\{X_s,\ s \le t\}$$

$$\mathscr{G}_t = \sigma\left\{Y_s,\ \int_0^s U_\theta d\theta,\ s \le t\right\}$$

$$\mathscr{A}_t = \mathscr{F}_t \vee \mathscr{G}_t.$$

Definition. An admissible control is a probability measure π on $(\Omega, \mathscr{G}_T)$ such that Y_t is a $\pi, \mathscr{G}_t$ Wiener process.

Let $\mathfrak{A}$ denote the set of all admissible controls π. Each $\pi \in \mathfrak{A}$ determines the joint distribution measure P_π of (X,Y,U) as follows. Given $Y \in C([0,T];R^1)$ and $U \in L^2([0,T];\mathscr{U})$ let $\overline{P}^{Y,U}$ be the unique probability measure on $(\Omega, \mathscr{F}_T)$ such that $\overline{P}^{Y,U}$ is the solution to the martingale problem [6] associated with (1.1)(a), and

$$\overline{P}^{Y,U}(X_0 \in B) = \int_B p_0(x)dx$$

for all Borel $B \subset R^N$. Let

$$\overset{\circ}{P}_\pi(dX,dY,dU) = \overline{P}^{Y,U}(dX)\pi(dY,dU),$$

and define P_π by

$$\frac{dP_\pi}{d\overset{\circ}{P}_\pi} = Z_T \tag{2.1}$$

with Z_T as in (1.3). It can be shown that there exist independent P_π Wiener processes W_t and $\tilde{W}_t$ such that (1.1) holds P_π-almost surely.

Let us write $E_\pi, \overset{\circ}{E}_\pi$ for expectations with respect to $P_\pi, \overset{\circ}{P}_\pi$ respectively. Then (1.2) becomes

$$J(\pi) = E_\pi\left\{\int_0^T F(t,X_t,U_t)dt + G(X_T)\right\}. \tag{2.2}$$

We make the following assumptions about F and G.

(A6) F,G are measurable. For fixed (t,x), $F(t,x,\cdot)$ is continuous and convex on $\mathscr{U}$. For some $C,\ m \ge 0$,

$$0 \le F(t,x,u) \le C(1+|x|)^m$$

$$0 \le G(x) \le C(1+|x|)^m.$$

In (A_5) we take $\ell \ge m$, which implies that $J(\pi) < \infty$.

Our result about existence of an optimal control is:

Theorem. There exists $\pi^* \in \mathfrak{A}$ such that $J(\pi^*) \le J(\pi)$ for all $\pi \in \mathfrak{A}$.

In §'s 3,4 we indicate the method of proof. A detailed proof will be given elsewhere.

The projection of any $\pi \in \mathfrak{A}$ under $(Y,U) \to Y$ is Wiener measure μ on $C([0,T];R^1)$. Let $\pi^Y(dU)$ be a regular conditional distribution for U given Y. We call π admissible in the strict sense if $\pi \in \mathfrak{A}$ and π^Y is a Dirac measure, concentrated at a point $U(Y) \in L^2([0,T];\mathscr{U})$, μ-almost surely. It can be shown that $J(\pi^*)$ equals the infimum of $J(\pi)$ among strict sense admissible controls; but it has not been shown that a strict sense optimal control exists. By admitting wider sense controls $\pi \in \mathfrak{A}$, we in effect allow the control U_t to depend on auxiliary randomizations in addition to the observations Y_s for $s \le t$.

3. The filtering equations.

Given trajectories Y and U for the observation and control processes, consider the elliptic partial differential operators associated with (1.1)(a):

$$L_t = \frac{1}{2}\sum_{i,j=1}^N a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b\cdot\nabla, \tag{3.1}$$

where $a = \sigma\sigma'$ and ∇ is the gradient in x. Let

$$\check{L}_t = L_t - Y_t\sum_{i=1}^N\left(\sum_{j=1}^N a_{ij}\frac{\partial h}{\partial x_j}\right)\frac{\partial}{\partial x_i}, \tag{3.2}$$

$$e(t,x) = \frac{1}{2}Y_t^2(a\nabla h,\nabla h) - Y_t\left(\frac{\partial h}{\partial t} + L_t h\right) - \frac{h^2}{2}. \tag{3.3}$$

Let $p(t,x)$ be the unique solution in $L^2([0,T];H^1) \cap C([0,T];R^n)$ to the partial differential equation

$$\frac{\partial p}{\partial t} = (\check{L}_t)^* p + e(t,x)p \tag{3.4}$$

with $p(0,x) = p_0(x)$. The following key formula can be proved. Given $\pi \in \mathfrak{A}$, then for every bounded continuous f

$$\int_{R^N} p(t,x)\exp[Y_t h(t,x)]f(x)dx = \overset{\circ}{E}_\pi[f(X_t)Z_t|\mathscr{G}_t], \tag{3.5}$$

π-almost surely. The proof involves the backward partial differential equation adjoint to (3.4), to whose solutions an appropriate version of the Feynman-Kac formula is applied.

Let

$$q(t,x) = p(t,x)\exp[Y_t h(t,x)]. \tag{3.6}$$

Equation (3.5) implies that $q(t,x)$ is the unnormalized conditional density of X_t given $\mathscr{G}_t$ (in other words, given past observations and controls Y_s, U_s for $s \le t$.) It can be shown that q satisfies the Zakai equation

$$\frac{\partial q}{\partial t} = (L_t)^* q + hq dY_t \tag{3.7}$$

with $q(0,x) = p_0(x)$. The conditional density of X_t given $\mathscr{G}_t$ is

$$\tilde{q}(t,x) = \frac{q(t,x)}{\int_{R_N} q dx}. \tag{3.8}$$

4. A separated control problem.

A well known idea is to introduce the conditional distribution of a partially observed state X_t as the "state" in a new "separated" control problem. This idea is the key to the classical separation principle for linear-quadratic problems [2, Chap. VI.11]. Similar ideas occur in the control of partially observed Markov chains [5] and of jump processes [4].

In the present context, we may take $p(t,\cdot)$ as the state at time t in a separated problem, since the conditional distributions of X_t are determined from $p(t,\cdot)$ through (3.6) and (3.8). The dynamics of the state process in the separated control problem are (3.4). Both e and the coefficients of L_t^Y depend on trajectories Y and U for the observation and control processes. Let

$$\tilde{\Omega} = C([0,T];R^1) \times L^2([0,T];\mathscr{U}).$$

For each $(Y,U) \in \tilde{\Omega}$, $p = p^{Y,U}$ is the unique solution to (3.4) with the given initial data $p(0,x) = p_0(x)$. In (3.6) we also write $q = q^{Y,U}$ for the unnormalized conditional density. From (2.1) and elementary properties of conditional expectations with respect to $\overset{o}{P}_\pi$ and $\mathscr{G}_t$, (2.2) can be rewritten as

$$J(\pi) = \int_{\tilde{\Omega}} \left[\int_0^T \int_{R^N} F(t,x,U_t) q^{Y,U}(t,x)\,dx\,dt + \int_{R^N} G(x) q^{Y,U}(T,x)\,dx \right] d\pi.$$

The separated problem is to show that there exists $\pi^* \in \mathfrak{A}$ minimizing (4.1). Once this is shown, the Theorem in §2 follows immediately.

The proof of existence of π^* proceeds as follows. Let π_n be any minimizing sequence in $\mathfrak{A}$. The sequence of probability measures π_n is tight, and hence a subsequence converges weakly to a limit π^*. Moreover, $\pi^* \in \mathfrak{A}$. Finally, it is shown that

$$J(\pi^*) \leq \underline{\lim}_{n\to\infty} J(\pi_n);$$

the proof depends on linearity of b and convexity of F in the control variable u (see assumptions (A_2), (A_6) in §2.) as well as results from PDE about continuous dependence on Y,U of solutions $p^{Y,U}$ to (3.4).

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